Economic Oscillations in a Multi-Country Growth Model with Free Trade and Tourism

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Abstract
This paper introduces exogenous time-dependent shocks to the multi-country growth model with tourism proposed by Zhang (2015). This study generalizes the model by making all the time-dependent parameters as time-dependent parameters. The model examines dynamic interactions among economic growth, structural change, international trade and tourist flows. It introduces endogenous tourism within a general dynamic equilibrium framework. The model integrates the three well-known Solow growth model, Oniki-Uzawa trade model, and the Uzawa two-sector model, and introduces tourist flows between national economies. We simulate the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks.

Keywords: trade pattern; international tourism; growth; business oscillations; inequality between countries

1 Introduction

This study studies fluctuations in the multi-country growth model with international trade, economic structures and international tourism proposed by Zhang (2015). The growth mechanism is based on Uzawa’s two-sector growth model (Uzawa, 1961, 1963; Galor, 1992; Mino, 1996; Cremers, 2006; Li and Lin, 2008; and Stockman, 2009). Zhang’s model is based on Oniki and Uzawa and others (for instance, Oniki and Uzawa, 1965; Frenkel and Razin, 1987; Nishimura and Shimomura, 2002; and Sorger, 2002). Most of trade models with endogenous wealth are either limited to two-country or small open economies. The model in this study is for any number of national economies, each one supplying internationally tradable goods and domestic services (which can be consumed by foreign tourists). It is a dynamic general equilibrium model, treating the global economy as an integrated whole. The unique feature of Zhang’s model is that it introduces tourism into a dynamic general equilibrium model. Tourism is special in that it converts some non-traded goods into tradable ones.
International tourism affects national economies in different ways. On the one hand, tourism attracts resources such as labor and capital from other sectors of the economy and, on the other hand, the income generated by tourism encourages development of other economic activities (Sinclair, 2002; Lee and Chang, 2008; Schubert et al., 2011; Seentanah, 2011; Sun, 2014). There are many studies in the literature of economics (e.g., Sinclair and Stabler, 1997; Hazari and Sgro, 2004; and Hazari and Lin, 2011). Chao et al. (2009) observe that the study of tourism has been mainly static. It is necessary to build dynamic models on microeconomic foundation. Another important issue is related to economic structural changes with tourism. As tourism uses national resources, development of tourism affects economic structure (e.g., Corden and Neary, 1982; Copeland, 1991; Oh, 2005; Zeng and Zhu, 2011). In order to fully understand possible effects of tourism on national economic development and economic structure, it is necessary to build a dynamic general equilibrium framework (Dwyer et al. 2004; Blake et al. 2006).

This study generalizes Zhang’s model by allowing all the time-independent parameters to be time-dependent. Zhang (1993). Economic fluctuations are commonly observed in empirical studies. There are a lot of theoretical and empirical research about mechanisms and phenomena of economic fluctuations. Zhang (1991, 2005, 2006) shows how modern dynamic analysis can be applied to different economic systems, identifying existence of cycles, regular as well as irregular oscillations, and chaos in economic systems. There are also studies which empirically test validity of theories (e.g., Lucas, 1977; Chatterjee and Ravikumar, 1992; Gabaix, 2011; Giovanni, et al. 2014; Stella, 2015). Nevertheless, there are only a few theoretical models which identify fluctuations due to dynamic interdependence between economic growth, trade and tourism. The rest of the paper is organized as follows. Section 2 defines the basic model. Section 3 shows how we solve the dynamics and simulates the motion of the global economy. Section 4 carries out comparative dynamic analysis to examine the impact of fluctuations in some parameters on the motion of the global economy. Section 5 concludes the study. The appendix proves the lemma in Section 3.

2. The Model

The system consists of any (finite) number of national economies. Each national economy is described by the Solow-Uzawa growth model to describe national economies. Most of the models in the neoclassical growth theory model are extensions and generalizations of the pioneering works of Solow (Solow, 1956; Diamond, 1965; Stiglitz, 1967; Burmeister and Dobell, 1970; Benhabib et al. 2000; Druegon and Venditti, 2001; Ortigueira and Santos, 2002). Uzawa (1961) extends Solow’s one-sector economy by a breakdown of the productive system into two sectors using capital and labor, one of which produces industrial goods, the other consumption goods. We follow the Oniki-Uzawa model (see also, Brecher, et al., 2002; Sorger, 2002) to describe trade patterns. The national economies produce a homogenous tradable commodity (see also Ikeda and Ono, 1992). Following Zhang (2015), we take account of tourism in our trade model. Domestic households consume both
industrial goods and services, while foreign tourists consume only services. Households own assets of
the economy and distribute their incomes to consume and save. Production sectors or firms use
capital and labor. Exchanges take place in perfectly competitive markets. Saving is undertaken only by
households. The system consists of multiple countries, indexed by \( j = 1, ..., J \). Each country has a
fixed labor force, \( N_j \), \( j = 1, ..., J \). Let \( K_j(t) \) and \( K^*_j(t) \) stand for respectively the capital stocks
employed and the wealth owned by country \( j \). We also introduce \( K^g_j(t) \) and \( K^s_j(t) \) to represent the
capital stocks employed by country \( j \’s \) industrial sector and service sector. Capital is both
internationally and domestically completely mobile. Labor is internationally immobile and
domestically completely mobile. We denote wage and interest rates by \( w_j(t) \) and \( r_j(t) \), respectively, in the \( j \)”th country. In the free trade system the interest rate is identical throughout the world
economy, i.e., \( r(t) = r_j(t) \). We use subscripts, \( i, s \), to denote the industrial and services sectors,
respectively. Let \( F_{q}(t) \) stand for the output levels of \( q \’s \) sector in region \( j \) at time \( t \), \( q = i, s \).

Behavior of producers
We assume that there are two productive factors, capital, \( K_q(t) \), and labor, \( N_q(t) \), at each point in
time \( t \). The production functions are specified as

\[
F_{q}(t) = A_{q}(t)K^\alpha_{q}(t)N^{\beta_{q}(t)}(t),
\]

\[
A_{q}(t), \alpha_{q}(t), \beta_{q}(t), \alpha_{q}(t) + \beta_{q}(t), = 1, \quad j = 1, ..., J, \quad q = i, s, \tag{1}
\]

where \( A_{q}(t), \alpha_{q}(t), \) and \( \beta_{q}(t) \) are time-dependent variables. It should be noted that in Zhang
these parameters are time-independent. We use \( p_{j}(t) \) to stand for country \( j \’s \) price of consumer
goods. The marginal conditions are

\[
r(t) + \delta(t) = \frac{\alpha_{q}(t)F_{q}(t)}{K_{q}(t)} = \frac{\alpha_{q}(t)p_{j}(t)F_{s}(t)}{K_{s}(t)}, \quad w_{j}(t) = \frac{\beta_{q}(t)N_{q}(t)}{F_{q}(t)} = \frac{\beta_{s}(t)p_{j}(t)F_{s}(t)}{N_{s}(t)}, \tag{2}
\]

where \( \delta(t) \) are the depreciation rate of physical capital in country \( j \).

Behavior of consumers
Consumers make decisions on choice of consumption level of commodity, how much to travel, as well
as on how much to save. This study uses the approach to consumers’ behavior proposed by Zhang
(1993). We use \( \bar{k}_j(t) \) to stand for the per capita wealth in country \( j \). The representative household
obtains the current income

\[
y_{j}(t) = r(t)\bar{k}_{j}(t) + w_{j}(t). \tag{3}
\]
We call \( y_j(t) \) the current income in the sense that it comes from consumers’ wages and current earnings from ownership of wealth. The disposable income is equal to

\[
\hat{y}_j(t) = y_j(t) + \bar{k}_j(t). \tag{4}
\]

The disposable income is used for saving and consumption. The value, \( \bar{k}_j(t) \), (i.e., \( p_i(t)K_j(t) \)), in the above equation is a flow variable. Under the assumption that selling wealth can be conducted instantaneously without any transaction cost, we may consider \( \bar{k}_j \) as the amount of the income that the consumer obtains at time \( t \) by selling all of his wealth. The consumer has the total amount of income equaling \( \hat{y}_j \) to spend on consumer goods, \( c_{sj}(t) \), capital goods, \( c_{ij}(t) \), tourist consumption in country \( q \), \( c_{jq}(t) \), and savings, \( s_j(t) \). The total cost for touring countries are

\[
\sum_{q \neq j}(t_{jq}(t) + p_q(t)c_{jq}(t))d_{jq}(t),
\]

where \( t_{jq}(t), d_{jq}(t), \) and \( p_q(t)c_{jq}(t) \) are respectively, the (fixed) transportation cost of each time from country \( j \) to country \( q \), the visit times from country \( j \) to country \( q \), and consumption of country \( q \)'s services by the tourist from country \( j \). For simplicity of analysis we neglect transportation costs, that is \( t_{jq} = 0 \). The budget constraints are

\[
c_q(t) + p_j(t)c_{sj}(t) + \sum_{q \neq j}p_q(t)d_{jq}(t) + s_j(t) = \hat{y}_j(t), \tag{5}
\]

where \( d_{jq}(t) = c_{jq}(t)d_{jq}(t) \). We assume that utility functions, \( U_j(t) \), are specified as follows

\[
U_j(t) = c_{ij}^{\xi_{ij}(t)}(t) c_{iq}^{\gamma_{iq}(t)}(t) s_{ij}^{\lambda_{ijj}(t)} (t) \prod_{q \neq j} d_{jq}^{\varepsilon_{ijj}(t)}(t), \quad \xi_{ij}(t), \gamma_{ij}(t), \lambda_{ij}(t) > 0, \varepsilon_{ijj}(t) \geq 0. \tag{6}
\]

Maximizing \( U_j \) subject to budget (6) yields

\[
c_{ij}(t) = \xi_{ij}(t)\hat{y}_j(t), \quad p_j(t)c_{sj}(t) = \gamma_{ij}(t)\hat{y}_j(t), \quad s_j(t) = \lambda_{ij}(t)\hat{y}_j(t), \quad p_q(t)d_{jq}(t) = \varepsilon_{jq}(t)\hat{y}_j(t), \quad q \neq j, \quad j, q = 1, \ldots, J, \tag{7}
\]
where
\[
\xi_j(t) = \rho_j(t) \xi_{0j}(t), \quad \gamma_j(t) = \rho_j(t) \gamma_{0j}(t), \quad \lambda_j(t) = \rho_j(t) \lambda_{0j}(t), \quad \epsilon_{jq}(t) = \rho_j(t) \epsilon_{0jq}(t),
\]
\[
\rho_j(t) = \frac{1}{\xi_{0j}(t) + \gamma_{0j}(t) + \lambda_{0j}(t) + \sum_{q \neq j} \epsilon_{0jq}(t)}.
\]

According to the definition of \( s_j(t) \), the wealth accumulation is given by
\[
\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t) - \frac{\dot{N}_j(t)}{N_j(t)} \bar{k}_j(t).
\]  

The equation simply states that the change in the wealth is equal to the savings minus the disavings.

**Full employment of capital and labor**

The total capital stocks utilized by country \( j \), \( K_j(t) \), is distributed between the two sectors. Full employment of labor and capital implies
\[
K_y(t) + K_g(t) = K_j(t), \quad N_y(t) + N_g(t) = N_j(t), \quad j = 1, ..., J.
\]  

**Balance conditions for global wealth**

The total capital stocks employed by the production sectors is equal to the total wealth owned by all the countries. That is
\[
\sum_{j=1}^{J} K_j(t) = \sum_{j=1}^{J} \bar{k}_j(t) N_j(t).
\]  

**Equilibrium conditions for national consumer goods**

For each country, the demand for services equals the supply of services at any point time
\[
N_j(t) c_j(t) + \sum_{q \neq j} d_{qj}(t) N_q(t) = F_j(t).
\]  

We thus built the dynamic growth model with endogenous distribution of wealth, consumption and labor distribution, and capital accumulation. We now examine dynamic properties of the system.
3 The Dynamics and Equilibrium

Although the economic system is nonlinear and highly dimensional, we can rely on computer simulation to plot the motion of the dynamic system. The following lemma provides a computational procedure for calculating all the variables at any point in time. First, we introduce a variable $z_i(t)$

$$z_i(t) \equiv \frac{r(t) + \delta_k}{w_i(t)}.$$

Lemma

The motion of the economic system is determined by $J$ differential equations with $z_i(t)$ and \( \{k_j(t)\} \), where \( \{k_j(t)\} \equiv (k_2(t), \ldots, k_J(t)) \) as the variables

$$z_i(t) = \Phi_j(z_i(t), \{k_j(t)\}),$$

$$\dot{k}_j(t) = \Phi_j(z_i(t), \{k_j(t)\}), \quad j = 2, \ldots, J,$$

in which $\Phi_j(t)$ are unique functions of $z_i(t)$ and \( \{k_j(t)\} \) defined in the appendix. At any point in time the other variables are uniquely determined as functions of $z_i(t)$ and \( \{k_j(t)\} \) by the following procedure: $r(t)$ by (A2) $\rightarrow w_j(t)$ by (A3) $\rightarrow p_j(t)$ by (A3) $\rightarrow k_j(t)$ by (A15) $\rightarrow \dot{y}_j(t)$ by (A8) $\rightarrow c_y(t)$, $c_y(t)$, $d_y(t)$ and $y(t)$ by (11) $\rightarrow N_y(t)$ by (A9) $\rightarrow N_y(t)$ by (A10) $\rightarrow K_y(t)$ and $K_y(t)$ by (A1) $\rightarrow F_y(t)$ by (9) $\rightarrow F_y(t)$ and $F_y(t)$ by the definitions $\rightarrow U_y(t)$ by (6) $\rightarrow K(t) = \sum_{j=1}^{J} K_j(t)$.

The lemma presents a computational procedure for plotting the motion of the economic system with any number of national economies. First, we simulate the model with the following time-independent parameter values

$$N_1 = 20, \quad N_2 = 30, \quad N_3 = 10, \quad A_1 = 1.2, \quad A_2 = 1, \quad A_3 = 0.8, \quad A_4 = 1.1, \quad A_5 = 0.9, \quad A_6 = 0.7,$$

$$\alpha_{i1} = 0.31, \quad \alpha_{i2} = 0.33, \quad \alpha_{i3} = 0.32, \quad \alpha_{s1} = 0.33, \quad \alpha_{s2} = 0.32, \quad \alpha_{s3} = 0.36, \quad \delta_{k1} = 0.05,$$

$$\delta_{k2} = 0.04, \quad \delta_{k3} = 0.045, \quad \lambda_{01} = 0.7, \quad \gamma_{01} = 0.15, \quad \xi_{01} = 0.1, \quad \epsilon_{012} = 0.002, \quad \epsilon_{012} = 0.007,$$

$$\lambda_{02} = 0.65, \quad \gamma_{02} = 0.07, \quad \xi_{02} = 0.12, \quad \epsilon_{021} = 0.004, \quad \epsilon_{023} = 0.008, \quad \lambda_{03} = 0.6, \quad \gamma_{03} = 0.98, \quad \xi_{03} = 0.15, \quad \epsilon_{031} = 0.004, \quad \epsilon_{032} = 0.008.$$  

(13)

This case is explained by Zhang (2015). We may interpret (13) as long-term trends. The next section examines what happen to the economic system if some parameters are fluctuated. The parameters, $\alpha_j$, in the Cobb-Douglas productions are often specified near 0.3 in the literature of
economic growth. It should be remarked that there are many empirical studies about income elasticity of tourism demand (Syriopoulos, 1995; Lanza et al., 2003), and price elasticities (Gaín-Múnos, 2007). We specify the initial conditions as follows:

\[ z_i(0) = 0.06, \quad k_2(0) = 3, \quad k_3(0) = 2. \]  \hspace{1cm} (14)

The motion of the system is given in Figure 1, in which each country's gross domestic product (GDP) and the world's gross global product (GGP) are defined as follows

\[ Y_j(t) = F_j(t) + p_j(t)F_{ij}(t), \quad Y(t) = \sum_{j=1}^{J} Y_j(t). \]

![Figure 1. The Motion of the Three National and Global Economies](image)

From Figure 1 we observe that all the variables become stationary in the long term. Following the lemma under (13), we calculate the equilibrium values of the variables as follows

\[ Y = 108.47, \quad K = 339.62, \quad r = 0.06, \quad Y_1 = 41.97, \quad Y_2 = 53.91, \quad Y_3 = 12.59, \quad K_1 = 121.26, \]

\[ K_2 = 177.22, \quad K_3 = 41.14, \quad F_{i1} = 28.69, \quad F_{i2} = 37.55, \quad F_{i3} = 5.89, \quad K_{i1} = 81.26, \]

\[ K_{i2} = 124.58, \quad K_{i3} = 18.03, \quad N_{i1} = 13.8, \quad N_{i2} = 20.8, \quad N_{i3} = 4.83, \quad F_{i1} = 12.62, \quad F_{i2} = 14.47, \]

\[ F_{i3} = 6.21, \quad K_{s1} = 40.01, \quad K_{s2} = 52.63, \quad K_{s3} = 23.1, \quad N_{s1} = 6.2, \quad N_{s2} = 9.2, \quad N_{s3} = 5.17, \]
\[ p_1 = 1.05, \quad p_2 = 1.13, \quad p_3 = 1.08, \quad w_1 = 1.44, \quad w_2 = 1.21, \quad w_3 = 0.83, \quad \bar{k}_1 = 8.56, \]
\[ \bar{k}_2 = 4.81, \quad \bar{k}_3 = 2.42, \quad c_{i1} = 1.22, \quad c_{i2} = 0.89, \quad c_{i3} = 0.6, \quad c_{s1} = 0.58, \quad c_{s2} = 0.46, \quad c_{s3} = 0.3, \]
\[ d_{12} = 0.022, \quad d_{13} = 0.079, \quad d_{21} = 0.028, \quad d_{23} = 0.055, \quad d_{31} = 0.015, \quad d_{32} = 0.028. \]

It is straightforward to calculate the values of the three eigenvalues as follows

\[ \{-0.243, -0.185, -0.147\}. \]

The eigenvalues are real and negative. The equilibrium point is locally stable. We now study what will happen to the global economy when some parameters experience exogenous fluctuations.

4 Comparative Dynamic Analysis

This section examines effects of changes in some parameters. We simulated the motion of the national and global economies under (13) and (14). Zhang (2015) shows how the system reacts to a once-for-all change in parameters. This section shows how the system reacts to time-dependent changes in parameters. For convenience we consider the parameters in (13) as the long-term average values. We make small perturbations around these long-term values. In this study we use \( \Delta x_j(t) \) to stand for the change rate of the variable \( x_j(t) \) due to changes in a parameter value.

Perturbations in the HPE’s total factor productivity of the capital goods sector

First, we study effects of the HPE’s technological change in the capital goods sector on the global economy. It has been argued that productivity differences explain much of the variation in incomes across countries, and technology plays a key role in determining productivity. We now show how fluctuations in the HPE’s total factor productivity of the capital goods sector affects trade patterns and national economies. We see what will happen to the global economy when

\[ A_{i1}(t) = 1.2 + 0.1 \sin(t). \]

We plot the effects in Figure 2. As the technology is perturbed, the global and national economies fluctuate. The global wealth and total product are slightly affected. The HPE’s output level, capital employed, and wealth experience strong fluctuations. The rate of interest and the wage rates of the PE and the LPE are slightly affected. The HPE’s wage rate and price of consumer goods fluctuate. The capital stocks and labor employed by the HPE’s capital sector and output level oscillate. The tourist patterns are also oscillatory. The macroeconomic variables fluctuate greatly and the microeconomic variables fluctuate slightly.
Fluctuations in the LPE’s population

Some theoretical and empirical studies on the relationship between population change and economic development show situation-dependent interactions between population and economic growth (e.g., Galor and Weil, 1999; Boucekkine, et al., 2002; Furuoka, 2009; Bretschger, 2013; and Yao et al., 2013). We now show how the global economy is affected when the LPE experiences the following fluctuation in population

\[ N_1(t) = 10 + 2 \sin(t). \]

We plot the effects in Figure 3. The LPE’s population fluctuations cause little changes in the other two economies’ macroeconomic and microeconomic variables. The LPE’s total output level and output levels of the two sectors fluctuate strongly. The rate of interest and wage rates in all the economies are slightly affected. We see that changes in the LPE’s population have little impact on the prices, wage rates and rate of interest, as the neoclassical economic theory predicts. The LPE’s labor and capital distributions oscillate. The frequencies that the household travels to the HPE and PE are oscillatory.
Figure 3. Fluctuations in the LPE’s Population

Fluctuations in the HPE’s propensity to tour the LPE
Preferences of different households are important for understanding economic structures. We now examine the effects of the following fluctuations in the HPE’s propensity to tour the LPE

\[ \varepsilon_{013}(t) = 0.007 + 0.003\sin(t). \]

The fluctuations lead to fluctuations in the tours by the HPE to the LPE. The LPE experiences fluctuations in the economic structure. There are oscillations in the LPE’s output levels, labor distribution, and capital distribution. We see that although the preference change has little impact on the global economy, it has strong impact on the macroeconomic variables of LPE. The rate of interest, prices and wage rates are slightly affected.
Figure 4. Fluctuations in the HPE’s propensity to tour the LPE

Fluctuations in the HPE’s propensity to save

We are concerned with effects of the following fluctuations of the HPE’s propensity to save

\[ \lambda_{01}(t) = 0.7 + 0.1 \sin(t). \]

The simulation result is plotted by Figure 5. As the household in the HPE oscillates the propensity to save, the wealth level of HPE's household fluctuates slightly. Similar to in the permanent income hypothesis, this conclusion implies that the household will adapt saving smoothly over time. The fluctuations have negligible impact on the macroeconomic variables. The HPE’s consumption levels of the goods and travels fluctuate.
Figure 5. Fluctuations in the HPE's Propensity to Save

Fluctuations of the LPE's propensity to consume consumer goods
We are concerned with the effects of the following fluctuations in the LPE's propensity to consume consumer goods

$$\gamma_{03}(t) = 0.08 + 0.04\sin(t).$$

The simulation result is plotted by Figure 6. The household in the LPE household’s consumption levels of the two goods, travels to the HPE and the PE fluctuate. The LPE’s capital and labor inputs largely oscillatory. Nevertheless, the LPE’s total capital employed and the total product change slightly. This implies that the effects of national fluctuations are limited to the nation itself and have weak impact on the global economy.
Figure 6. Fluctuations the LPE’s Propensity to Consume Consumer Goods

5 Concluding Remarks

This paper introduced exogenous time-dependent shocks to the multi-country growth model with tourism proposed by Zhang (2015). This study generalized Zhang’s model by making all the time-dependent parameters as time-dependent parameters. The model examines dynamic interactions between economic growth, structural change, international trade and tourist flows. It introduces endogenous tourism in a general equilibrium trade model with endogenous wealth accumulation and international trade. The model integrates the three well-known Solow growth model, Oniki-Uzawa trade model, and the Uzawa two-sector model, and introduces tourist flows between national economies. We demonstrated that the motion of the $J$-country world economy can be described by $J$ differential equations. We also simulated the global economy with three countries. We simulated the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks. We showed that the world dynamics has a unique equilibrium. We carried out comparative dynamic analysis with regard to fluctuations in one country’s total productivity factor, the propensity to save, the propensity to tour other countries, and the population. It should also be remarked that it is possible to extend and generalize in different directions. For instance, we may take other forms of the utility functions or/and production functions. The Solow model and Uzawa two-sector growth models are the two key models in the neoclassical economic growth theory and the Oniki-Uzawa growth model is a main key model of global economic dynamics with capital accumulation. These models have been generalized and extended in different ways. We may extend our model on the basis of the contemporary literature of economics.
Appendix: Proving the Lemma

The appendix confirms the lemma. We omit time in expressions wherever there is no confusion. From (2), we have

\[
z_j = \frac{r + \delta_k}{w_j} = \frac{a_j N_{ij}}{K_{ij}} = \frac{b_j N_{sj}}{K_{sj}},
\]  

(A1)

where

\[
a_j = \frac{\alpha_{ij}}{\beta_{ij}}, \quad b_j = \frac{\alpha_{sj}}{\beta_{sj}}.
\]

From \( z_j / a_j = N_{ij} / K_{ij} \) and (2)

\[
r(z_j) = \frac{\alpha_{ij} A_{ij}}{a_j} z_j^{\beta_i} - \delta_{ij}, \quad j = 1, \ldots, J.
\]  

(A2)

From (A2) we get

\[
z_j(z_1) = a_j \left( \frac{r + \delta_{kj}}{\alpha_{ij} A_{ij}} \right)^{1/\beta_i}, \quad j = 2, \ldots, J.
\]  

(A3)

From (A1) and (A2)

\[
w_j(z_1) = \frac{r + \delta_{kj}}{z_j}.
\]  

(A4)

From \( z_j = b_j N_{sj} / K_{sj} \) and (1), we get

\[
p_j(z_1) = \frac{b_j^{\beta_r} \left( r + \delta_{kj} \right)}{\alpha_{sj} A_{sj} z_j^{\beta_r}}.
\]  

(A5)

From (11) and (7) we get
\[ \gamma_j \hat{y}_j N_j + \sum_{q \neq j} \varepsilon_{aq} \hat{y}_q N_q = p_j F_{sj}. \]  
(A6)

Inserting (1) in (A6) yields

\[ \gamma_j \hat{y}_j N_j + \sum_{q \neq j} \varepsilon_{aq} \hat{y}_q N_q = \frac{w_j N_{sj}}{\beta_{sj}}. \]  
(A7)

By (3) we get

\[ \hat{y}_j(z_1, \bar{k}_j) = (1 + r)\bar{k}_j + w_j. \]  
(A8)

Substituting (A8) into (A7) yields

\[ N_{sj}(z_1, (\bar{k}_j), t) = \bar{R}_j + \left( \gamma_j N_j \bar{k}_j + \sum_{q \neq j} \varepsilon_{aq} N_q \bar{k}_q \right) R_j, \]  
(A9)

where

\[ R_j(z_1, t) = \frac{1 + r}{w_j} \beta_{sj}, \quad \bar{R}_j(z_1, t) = \beta_{sj} \gamma_j N_j + \frac{\beta_{sj}}{w_j} \sum_{q \neq j} w_q \varepsilon_{aq} N_q. \]

From \( N_{sj} + N_{sj} = N_j \) and (A9), we solve

\[ N_{sj}(z_1, (\bar{k}_j), t) = N_j - N_{sj}. \]  
(A10)

From (A10) and (A11) we have the labor distribution as functions of \( z_1, (\bar{k}_j), \) and \( t \). From (A1) and (A10) we get

\[ K_j(z_1, (\bar{k}_j), t) = \frac{a_j N_{sj}}{z_j}, \quad K_j(z_1, (\bar{k}_j), t) = \frac{b_j N_{sj}}{z_j}. \]  
(A11)

From (9) and (10), we have

\[ \sum_{j=1}^{J} (K_j + K_j) = \sum_{j=1}^{J} \bar{k}_j N_j. \]  
(A12)
Inserting (A12) in (A11) implies

\[
\sum_{j=1}^{J} \alpha_j N_j \frac{a_j}{z_j} + \sum_{j=1}^{J} \bar{z}_j N_j = \sum_{j=1}^{J} \bar{k}_j N_j. \tag{A13}
\]

where

\[
\bar{z}_j \equiv \frac{(b_j - a_j)}{z_j}.
\]

Insert (A9) in (A13)

\[
\gamma_1 N_1 \bar{z}_1 R_1 \bar{k}_1 + \sum_{j=1}^{J} \left( \sum_{q \neq j}^{J} \varepsilon_{qj} N_q \bar{k}_q \right) \bar{z}_j R_j = R_0 + \bar{k}_1 N_1. \tag{A14}
\]

where

\[
R_0(z_1, \{\bar{k}_j\}, t) = \sum_{j=2}^{J} \bar{k}_j N_j - \sum_{j=1}^{J} \frac{a_j N_j}{z_j} - \sum_{j=2}^{J} \bar{z}_j R_j - \sum_{j=2}^{J} \gamma_j N_j \bar{z}_j R_j \bar{k}_j.
\]

where \(\{\bar{k}_j\} \equiv (\bar{k}_2, \ldots, \bar{k}_J)\). Solve (A14) with respect to \(\bar{k}_1\)

\[
\bar{k}_1 = \Phi(z_1, \{\bar{k}_j\}, t) = \frac{1}{N_1} \left[ R_0 \left( \sum_{q=2}^{J} \varepsilon_{qj} N_q \bar{k}_q \right) \bar{z}_1 R_1 - \sum_{j=2}^{J} \left( \sum_{q \neq j}^{J} \varepsilon_{qj} N_q \bar{k}_q \right) \bar{z}_j R_j \right] \left( \gamma_1 \bar{z}_1 R_1 - 1 + \sum_{j=2}^{J} \varepsilon_{ij} \bar{z}_j R_j \right)^{-1} . \tag{A15}
\]

Substitute \(s_j = \lambda \tilde{y}_j\) and \(r \bar{k}_j + w_j\) into (8)

\[
\tilde{k}_1 = \Phi_0(z_1, \{\bar{k}_j\}, t) \equiv (1 + r) \tilde{\lambda} \Phi + \tilde{\lambda} w_1 - \frac{\tilde{k}_1 \dot{N}_1}{N_1}, \tag{A16}
\]

\[
\tilde{k}_j = \Phi_j(z_1, \bar{k}_j, t) \equiv (1 + r) \tilde{\lambda} \bar{k}_j + \tilde{\lambda} w_j - \frac{\bar{k}_j \dot{N}_1}{N_j}, \quad j = 2, \ldots, J. \tag{A17}
\]

Taking derivatives of equation (A15) with respect to \(r\) yields
\[
\dot{k}_1 = \frac{\partial \Phi}{\partial z_1} \dot{z}_1 + \frac{\partial \Phi}{\partial t} + \sum_{j=2}^{J} \Phi_j \frac{\partial \Phi}{\partial k_j},
\]
(A18)

where we use (A17). Insert (A18) in (A16)

\[
\dot{z}_1 = \Phi_1(z_1, \{k_j\}, t) = \left( \Phi_0 - \frac{\partial \Phi}{\partial t} - \sum_{j=2}^{J} \Phi_j \frac{\partial \Phi}{\partial k_j} \right) \left( \frac{\partial \Phi}{\partial z_1} \right)^{-1}.
\]
(A19)

We thus confirmed the lemma. The proof also provides a computational procedure.

REFERENCES